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The reflexion of internal/inertial waves from bumpy surfaces

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When internal and/or inertial waves reflect from a smooth surface which is not plane, there is in general some energy flux which is 'back-reflected' in the opposite direction to that of the incident energy flux (in addition to that 'transmitted' in the direction of the reflected rays), provided only that the incident wavelength is sufficiently large in comparison with the length scales of the reflecting surface. The reflected wave motion due to an incident plane wave is governed by a Fredholm integral equation whose kernel depends on the form of the reflecting surface. Some specific examples are discussed, and the special case of the 'linearized boundary' is considered in detail. For an incoming plane wave incident on a sinusoidally varying surface of sufficiently small amplitude, in addition to the main reflected wave two new waves are generated whose wave-numbers are the sum and difference respectively of those of the surface perturbations and the incident wave. If the incident wave-number is the smaller, the difference wave is back-reflected.

1. Introduction and summary

Internal and inertial waves are the means by which small disturbances propagate through density-stratified fluids and homogeneous rotating fluids. Their fundamental properties are well known and are described in, for example, Phillips (1966) and Greenspan (1968). The reflexion of internal or inertial waves from infinite plane surfaces was first discussed in detail by Phillips (1963) (although it is mentioned briefly by Eckart (1960)), and the results for an inviscid fluid are summarized in the monograph by Greenspan (1968). Briefly, a plane wave incident on such a surface is reflected as another plane wave whose crests make the same angle to the vertical, but opposite in sign, as those of the incident wave. The reflected wave-number and energy flux are modified accordingly.

In this paper we consider what happens when plane waves are incident on surfaces which are not flat. The analysis is two-dimensional (i.e. independent of one horizontal co-ordinate), and it is assumed that the surface is smooth, so that it has a tangent plane at all points.† For plane waves incident from a given direction we may distinguish between two types of reflecting surface (see figure 1),

† The case of two plane surfaces intersecting in a sharp corner has been discussed by Hurley (1970).

namely (i) 'flat bump' topography, where the incident wave 'lights' or 'sees' the entire surface and the wave characteristics all reflect off in the same direction, and (ii) 'steep bump' topography, where some of the surface is sheltered so that some diffraction must take place, or the wave characteristics reflect off in both directions. Here we shall restrict consideration to flat bumps, and except for the last section assume that the topography is localized, in that the surface becomes planar at large distances from the bump.

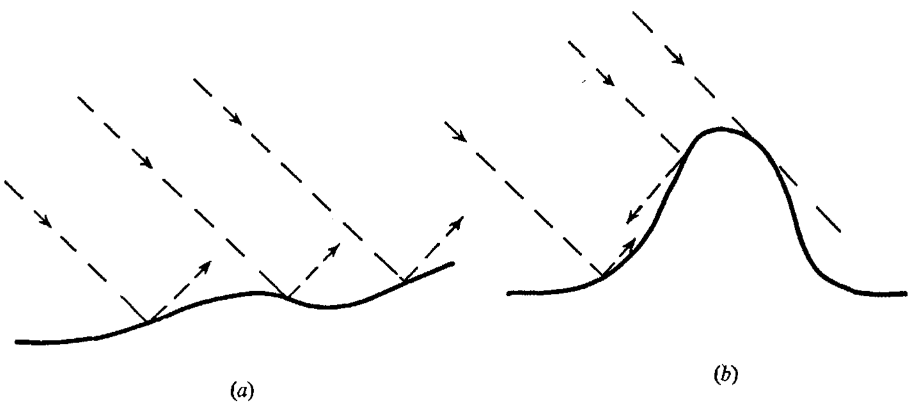


FIGURE 1 (*a*). 'Flat bump' topography, with the wave characteristics of the incident wave and their reflections shown, and (*b*) 'steep bump' topography.

One of the salient properties of internal/inertial waves is that, for motion of a single given frequency, the wave crests and energy flux vectors may only lie along two lines, which make equal and opposite angles with the vertical, so that the energy flux associated with this frequency may be in only four possible directions. This is a consequence of equation (2.3), and stands in contrast to waves which are governed by the common wave equation for non-dispersive waves

$$\nabla^2\phi = \frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2}, \tag{1.1}$$

where ϕ is some field variable, t is time and c_0 is the wave speed. For wave motion of this type with a given frequency the spatial structure is determined by a Helmholtz equation, and the energy flux may in principle be in any direction at all. From these considerations we might expect the phenomena of reflexion of these two kinds of waves from a 'flat' bumpy surface to differ along the lines indicated in figure 2. For the internal/inertial wave case only a 'transmitted' and a 'back-reflected' wave are possible, whereas in the common case the bump acts as the source of a cylindrical type (for two space dimensions) scattered wave which is present as well as the main reflected (or transmitted, according to view-point) wave. For a discussion of this type of reflexion with various boundary conditions appropriate to e.m., sound and elastic waves the reader is referred to several hundred pages of Morse & Feshbach (1953).

The plan of the paper is as follows—in §§2 and 3 the basic equations and assumptions are stated and a suitable analytic expression for the radiation condition is derived. Applying this radiation condition in §4 to both the back-reflected and the transmitted wave and incorporating the boundary condition leads to a Fredholm integral equation of the second kind which determines the wave field. It is also shown that the solution to this equation is consistent with the WKBJ approximation provided that a radius of curvature may be defined at each point of the surface (i.e. the equation for the surface has a second derivative

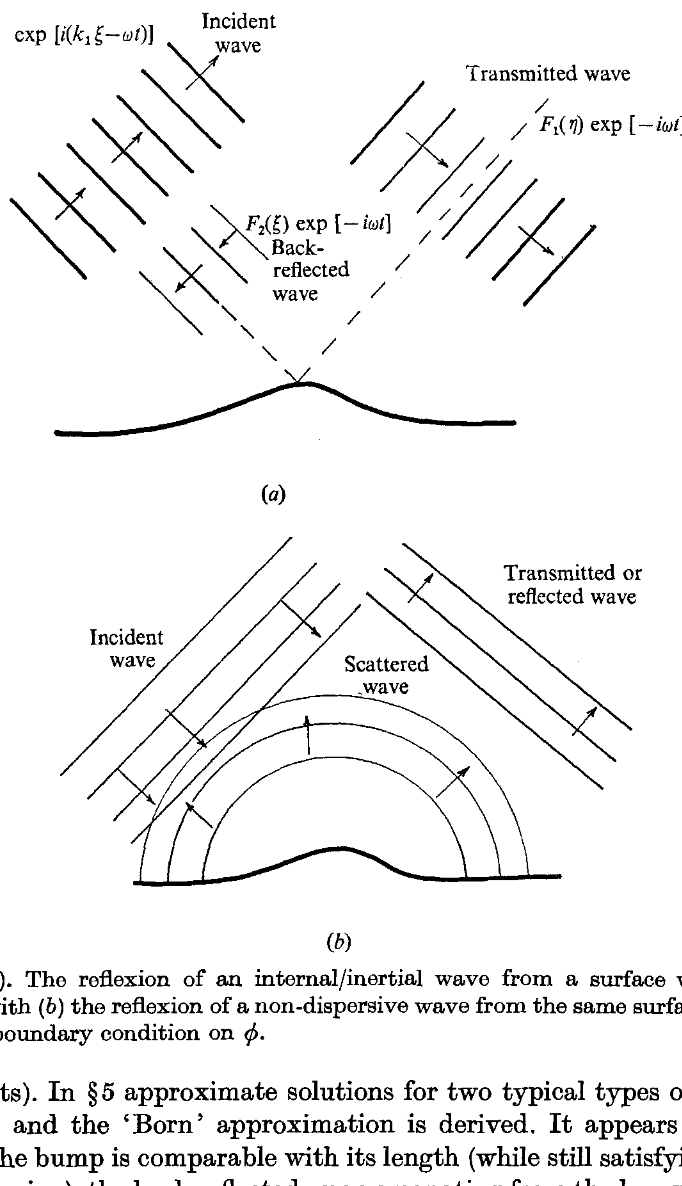


FIGURE 2(*a*). The reflexion of an internal/inertial wave from a surface with a bump, compared with (*b*) the reflexion of a non-dispersive wave from the same surface, with some prescribed boundary condition on ϕ.

at all points). In §5 approximate solutions for two typical types of bumps are considered and the 'Born' approximation is derived. It appears that, if the height of the bump is comparable with its length (while still satisfying the 'flat-bump' criterion), the back reflected wave emanating from the bump may be not much smaller than the transmitted wave emanating from the same part of the surface. On the other hand, for some shapes and incident wave combinations it may vanish entirely.

In § 6 it is shown that if the deviations of the surface from a flat plane are small in relation to the incident wavelength and the length scale of the bumps, an approximate solution for the wave motion may be comparatively easily obtained. This is the solution which would be obtained by linearizing the boundary condition in the usual way (e.g. as is done for surface waves). When an incident plane wave interacts with a single Fourier component of the bottom topography, two new waves with sum and difference wave-numbers are produced, to first order. The sum wave is always in the transmitted direction, whereas the difference wave is back-reflected if the incident wavelength is sufficiently long. Solutions for general linearized bumps may be readily obtained by Fourier superposition, and if one should wish to obtain the reflected motion for an arbitrary piece of bottom topography, the best procedure may be to solve the integral equation for the large bumps and then use the linearized approximation for the small ones.

Concerning the radiation condition, it should be noted that it is not satisfied by the work of Barcilon & Bleistein (1969*a*, *b*) where they considered the reflexion and diffraction of inertial waves from cylinders. Of the infinity of possible solutions which satisfy the surface boundary conditions, theirs are the ones which give no back-reflected wave motion, in the sense described above, and do not constitute the reflexion/diffraction pattern due to a single incident plane wave, as purported. The radiation condition is also not satisfied by the flow envisaged by Longuet-Higgins (1969) in his calculations of reflexion and transmission coefficients for various rough surfaces on a geometrical basis, and consequently these coefficients must also be invalid, aside from any considerations of sharp corners, viscous boundary layers and the like.

Throughout this paper the terms 'surface' and 'bottom topography' are used interchangeably because this work has been done with potential applications to the ocean in mind. Furthermore, corresponding results to those obtained here may be inferred for the case of a horizontal channel, where the fluid has a rigid upper boundary.

2. Basic equations

We consider the motion of an incompressible inviscid rotating stratified fluid, and take Cartesian axes x, y, z, z increasing vertically upward, with corresponding velocity components u, v, w. We take the axis of rotation to be vertical, and the linearized equations of motion in the rotating frame are

$$\left.\begin{aligned} \frac{\partial \mathbf{u}}{\partial t}+\mathbf{f}\times\mathbf{u} &= -\frac{1}{\rho_0(z)}\nabla p-\frac{\rho g\hat{\mathbf{z}}}{\rho_0(z)}, \\ \nabla.\mathbf{u} &= 0, \\ \frac{\partial \rho}{\partial t}+w\frac{d\rho_0}{dz} &= 0, \end{aligned}\right\} \tag{2.1}$$

where $\rho_0(z)$ is the equilibrium density, p and ρ are the perturbation pressure and density respectively, $\hat{\mathbf{z}}$ is the unit vector in the direction of z increasing, t is the time variable, $\mathbf{u}$ is the fluid velocity, g the acceleration due to gravity, and $\mathbf{f} = f\hat{\mathbf{z}} = 2\mathbf{\Omega}$ where $\mathbf{\Omega}$ is the angular velocity of the system. We next assume that the bottom topography and the incident wave motion are independent of the y co-ordinate, so that we may define a stream function $\psi(x, z, t)$ by the equations

$$u = -\partial\psi/\partial z, \quad w = \partial\psi/\partial x. \tag{2.2}$$

Equations (2.1) then yield the equation for ψ

$$\partial^2\nabla^2\psi/\partial t^2 + N^2\psi_{xx} + f^2\psi_{zz} = 0, \tag{2.3}$$

where N is the Brunt–Väisälä frequency defined by

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_0}{dz}. \tag{2.4}$$

If we further assume that all the fluid motion has the time dependence $e^{-i\omega t}$, then writing

$$\psi = \hat{\psi}(x, z)\, e^{-i\omega t}, \tag{2.5}$$

we obtain

$$\hat{\psi}_{xx} - c^2\hat{\psi}_{zz} = 0, \quad c^2 = \frac{\omega^2 - f^2}{N^2 - \omega^2}, \tag{2.6}$$

where the suffices denote derivatives. In order to have internal and/or inertial waves we require $c^2 > 0$, and for the sake of definiteness we will take

$$0 < f < \omega < N,$$

which is the case of greatest relevance for the ocean. We also assume that N^2 is constant. The conclusions of the following theory will still be valid in cases where $N^2(z)$ is not constant, however, provided only that N^2 be effectively constant in the regions of the fluid near the bottom topography. With N^2 constant, c^2 is constant and (2.6) has the general solution

$$\hat{\psi} = f(\xi) + g(\eta), \tag{2.7}$$

where f and g are arbitrary complex-valued functions of the real characteristic variables $\xi = z + cx$, $\eta = z - cx$.

We consider fluids of effectively infinite depth with a bottom surface or topography which has the equation

$$z = h(x), \tag{2.8}$$

where $h(x)$ is a differentiable function everywhere so that the bottom is smooth, $|h'(x)| < c$ so that the condition for 'flat' bumps is satisfied, and

$$h'(x) \to \begin{matrix}\alpha_L \\ \alpha_R\end{matrix} \quad \text{as} \quad x \to \begin{matrix}-\infty \\ +\infty\end{matrix}, \tag{2.9}$$

where α_L, α_R are constants, so that the bottom topography variations are localized. This latter assumption is not crucial for the theory, and is made for the sake of simplicity.† The condition for 'flat' bumps implies that there is a one–one correspondence between the ξ characteristics and the η characteristics, so (2.8) may be written in either of the two forms

$$\xi = -K(\eta), \quad \eta = -H(\xi), \tag{2.10}$$

† There is no real point in treating infinite sinusoidal bottom topography here, as such topography is not commonly met with in nature and further, Fourier superposition is not possible, since we are not restricting consideration to small bumps. See §6.

where the functions H, K are defined by (2.8) and will be monotonically increasing functions of their arguments. The boundary condition to be satisfied on this bottom surface is

$$\psi = 0. \tag{2.11}$$

We next consider a plane internal/inertial wave incident on this bottom topography, which we denote as

$$\psi_i = \epsilon \exp[i(k_1\xi - \omega t)], \tag{2.12}$$

where ϵ denotes the amplitude, sufficiently small so as to make the linearization of the basic equations valid throughout the following analysis. This has phase-propagation in the direction of increasing ξ with a downward group velocity (along the lines of constant ξ). A solution which satisfies the boundary conditions and appears at first sight to be the reflected wave due to ψ_i is

$$\psi_T = -\epsilon \exp[-i(k_1 K(\eta) + \omega t)], \tag{2.13}$$

so that

$$\psi_i + \psi_T = 0$$

on the surface $\xi = -K(\eta)$. Since $K(\eta)$ is monotonically increasing with η, the motion denoted by ψ_T has phase propagation in the direction of η decreasing. This suggests that the energy flux associated with the motion ψ_T above is outward away from the boundary, by analogy with plane waves. However, if one expresses ψ_T in terms of its Fourier integral of plane waves, one finds that, for almost every function $K(\eta)$ which satisfies the above conditions, ψ_T contains some plane waves with their phase propagation in the *reversed direction* (as the reader may readily see by inspecting the examples considered in §5). This implies in turn that the group velocity of these plane waves is directed downward, signifying that ψ_T contains some *incoming energy* from infinity (Lighthill 1965). The solution $\psi_i + \psi_T$, as expressed by equations (2.13) and (2.12), is a possible field of motion but it is not the motion generated by the single incoming wave ψ_i; some other additional sources of energy at infinity are required to create it. In order to determine the field of motion due to a single incoming plane wave we need to employ the appropriate radiation condition, and this is outlined in the next section.

3. The radiation condition

We consider a field of wave motion of a single frequency which is a function of one characteristic variable only, e.g.

$$\psi = F(\eta)\, e^{-i\omega t}, \tag{3.1}$$

and seek a general condition of $F(\eta)$ which implies that the energy flux associated with this wave field be in *one direction only*. $F(\eta)$ may be written

$$F(\eta) = \frac{1}{2\pi}\int_0^\infty \exp[-ik\eta]\,dk \int_{-\infty}^\infty F(\eta')\exp[ik\eta']\,d\eta' + \frac{1}{2\pi}\int_0^\infty \exp[ik\eta]\,dk \int_{-\infty}^\infty F(\eta')\exp[-ik\eta']\,d\eta', \tag{3.2}$$

which is effectively its Fourier integral representation. The first term consists of waves of the form $\exp[-i(k\eta+\omega t)]$, which have their group velocity directed upward and to the right, while the second term consists of waves of the form $\exp[i(k\eta-\omega t)]$, with group velocity directed downward and to the left. Now it has been shown by Lighthill (1965, 1967) that, for a periodic source of wave motion in a rotating stratified fluid the physically relevant solution is composed of plane waves whose group velocity is directed away from the source. No plane waves with incoming group velocity may be present unless an additional appropriate source is specified. An initial-value problem by Baines (1969) further demonstrates the validity of this result for the corresponding case of internal wave motion in a horizontal channel. Accordingly, if we require that the motion represented by (3.1) have no energy source at infinity 'upward and to the right', so that it contains only constituent plane waves of the form $\exp[-i(k\eta+\omega t)]$, then it is necessary and sufficient that $F(\eta)$ satisfy

$$F(\eta) = \frac{1}{2\pi}\int_0^\infty \exp[-ik\eta]\,dk\int_{-\infty}^\infty F(\eta')\exp[ik\eta']\,d\eta'. \tag{3.3}$$

Reversing the order of integration then yields

$$F(\eta) = \frac{1}{2\pi}\int_{-\infty}^\infty F(\eta')\,d\eta'\int_{-\infty}^\infty H_{ev}(k)\exp[ik(\eta'-\eta)]\,dk, \tag{3.4}$$

where $H_{ev}(k)$ is Heaviside's step function, whose Fourier transform (Lighthill 1958, p. 43) yields

$$\int_{-\infty}^\infty H_{ev}(k)\exp[ik(\eta'-\eta)]\,dk = 2\pi\left[\tfrac{1}{2}\delta(\eta'-\eta)+\frac{1}{2\pi i(\eta'-\eta)}\right]. \tag{3.5}$$

Hence (3.4) becomes

$$F(\eta) = \frac{i}{\pi}\,\mathrm{P}\int_{-\infty}^\infty \frac{F(\eta')}{\eta'-\eta}\,d\eta', \tag{3.6}$$

where P denotes that the principal value of the integral is to be taken. This is the required expression for the radiation condition. If it were specified instead that the energy flux be in the opposite direction, then the sign in front of the integral in (3.6) would be reversed.

This radiation condition is equivalent to that used by Cox & Sandstrom (1962) in their study of the generation of internal tides. Solutions subsequently presented by Sandstrom (1966) for a horizontal channel do not, however, satisfy this radiation condition and have the same characteristics as the motion described by equation (2.13). This work has been criticized by Baines (1969), where the equation corresponding to equation (3.6) here is derived for a horizontal channel. Other studies by Rattray, Dworski & Kovala (1969), Larsen (1969) and Robinson (1969) employ the correct radiation condition.

Longuet-Higgins (1969), following the work of Sandstrom (1966), has calculated the reflexion and transmission coefficients for a plane wave incident on periodic bottom topography of saw-tooth, square-wave and sinusoidal type, on a geometrical basis centred around the direction of reflexion of the wave characteristics from the surface. For the case of 'flat' bumps considered in the present paper, such transmission coefficients calculated in this way would always be unity, and the reflexion coefficients zero. However, as is shown below, the reflexion coefficient for flat bumps is in general non-zero, and may in fact be quite considerable. It therefore seems obvious that this simple basis is inadequate in general for the calculation of reflexion and transmission coefficients, to say nothing of the 'steep' bump case where the effects of diffraction must be considered.

Barcilon & Bleistein (1969*a*, *b*) in an approach which is apparently quite independent of that of any of the authors mentioned above, have investigated the reflexion and diffraction of inertial waves from a semi-infinite flat strip and smooth convex cylinders in general. However, with the exception of the work on a flat strip, the functions used to construct the solutions for the reflected and diffracted waves do not satisfy the radiation condition when expressed in terms of plane waves, even though their phase propagation is in the same direction at each point. Their solutions are in fact analogous to (2.13) of the previous section. Consequently these solutions must be incorrect, in that they do not represent the diffracted wave pattern due to a single incident plane wave.†

4. The integral equation and some general results

Using the radiation condition derived in the previous section, we now derive an equation which governs the reflected wave field from the 'flat' bump bottom topography. To this end we write

$$\left.\begin{aligned} \psi_i &= \epsilon \exp\left[i(k_1\xi - \omega t)\right] \quad \text{the incoming wave,} \\ \psi_T &= -\epsilon F_1(\eta)\exp\left[-i\omega t\right], \quad \text{the transmitted wave,} \\ \psi_R &= \epsilon F_2(\xi)\exp\left[-i\omega t\right], \quad \text{the reflected wave,} \end{aligned}\right\} \tag{4.1}$$

(see figure 2(*a*)), where $F_1(\eta)$ and $F_2(\xi)$ are complex-valued functions. The most general wave field possible is $\psi = \psi_i + \psi_T + \psi_R$, and on the boundary

$$\xi = -K(\eta), \quad \text{or} \quad \eta = -H(\xi), \tag{4.2}$$

we require $\psi = 0$, so that

$$\exp\left[ik_1\xi\right] - F_1(\eta) + F_2(\xi) = 0, \tag{4.3}$$

when (4.2) holds. The radiation condition (3.6) applied to each of ψ_T and ψ_R separately yields

$$F_1(\eta) = \frac{i}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{F_1(\eta')\,d\eta'}{\eta' - \eta}, \tag{4.4}$$

$$F_2(\xi) = \frac{i}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{F_2(\xi')}{\xi' - \xi}\,d\xi'. \tag{4.5}$$

The four equations (4.2)–(4.5) constitute a closed set which we wish to solve for the functions $F_1(\eta)$, $F_2(\xi)$.

If we define

$$\mathscr{F}_2(\eta) = F_2[-K(\eta)] = F_2(\xi), \tag{4.6}$$

† They may, however, be valid in the 'geometrical optics' (very high incident wavenumber) limit (see § 4).

where ξ and η are related by (4.2), then (4.5) becomes

$$\mathscr{F}_2(\eta) = -\frac{i}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{\mathscr{F}_2(\eta')\,(dK/d\eta')\,d\eta'}{K(\eta')-K(\eta)}. \tag{4.7}$$

Also, if we substitute $F_1(\eta)$ from (4.3) in (4.4) utilizing (4.2), (4.6), we obtain

$$\mathscr{F}_2(\eta) = -\exp[-ik_1K(\eta)] + \frac{i}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{\exp[-ik_1K(\eta')]}{\eta'-\eta}\,d\eta' + \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\mathscr{F}_2(\eta')\,d\eta'}{\eta'-\eta}. \tag{4.8}$$

If we define

$$G(\eta) = \frac{i}{2\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{\exp[-ik_1K(\eta')]}{\eta'-\eta}\,d\eta' - \tfrac{1}{2}\exp[-ik_1K(\eta)], \tag{4.9}$$

then adding (4.7) and (4.8) yields

$$\mathscr{F}_2(\eta) = G(\eta) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathscr{F}_2(\eta')\,\frac{d}{d\eta'}\log\left[\frac{K(\eta')-K(\eta)}{\eta'-\eta}\right]d\eta'. \tag{4.10}$$

This is the basic integral equation which must be solved in order to determine the wave motion, and is a *Fredholm equation of the second kind.* The kernel is non-singular since $K(\eta)$ is assumed to be differentiable, the principal-value singularity in (4.7) and (4.8) having been subtracted out. If we assume that $1/2\pi i$ is not an eigenvalue of (4.10), there will be a unique solution for $\mathscr{F}_2(\eta)$, and we shall proceed on this assumption. $F_1(\eta)$, $F_2(\xi)$ will then be given by

$$\left.\begin{aligned} F_1(\eta) &= \exp[-ik_1K(\eta)] + \mathscr{F}_2(\eta),\\ F_2(\xi) &= \mathscr{F}_2[-H(\xi)]. \end{aligned}\right\} \tag{4.11}$$

The kernel in (4.10) depends solely on the bottom topography, while the function $G(\eta)$ depends on the bottom topography and the incident wave-number k_1. $G(\eta)$ will vanish if the function $\exp[-ik_1K(\eta)]$ satisfies (3.6), which in general it will not, as the reader may readily verify. $F_2(\xi) = \mathscr{F}_2(\eta)$, as determined from (4.10), (4.11) is a 'back-reflected' wave whose energy flux is in the opposite direction to that associated with the incoming plane wave. It also gives the correction to the function $\exp[-ik_1K(\eta)]$ for the onward-transmitted wave. This back-reflected wave is perhaps the most significant feature of the present analysis (it is absent from the theory of Barcilon & Bleistein), and it is not detected by 'ray-theory' based on the WKB approximation (e.g. Keller & Mow 1969).

Before considering the possibilities of solving (4.10), it is instructive to consider the behaviour of $G(\eta)$ as k_1 becomes very large, implying that the incident wave-length is much less than length scales associated with the bottom topography. Utilizing (4.2) we may write

$$G(\eta) = \mathscr{G}(\xi) = -\tfrac{1}{2}\exp[ik_1\xi] - \frac{i}{2\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{\exp[ik_1\xi']H'(\xi')\,d\xi'}{H(\xi')-H(\xi)}, \tag{4.12}$$

where $H'(\xi) = dH/d\xi$. Furthermore,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{\exp[ik_1\xi']}{\xi'-\xi}\,d\xi' = i\pi\exp[ik_1\xi], \tag{4.13}$$

as may be seen from the Plemelj formulae of complex analysis (see, for example, Carrier, Krook & Pearson 1966), or the Fourier transform of $1/(\xi'-\xi)$ considered as a generalized function (Lighthill 1958). Hence we have

$$G(\eta) = \mathscr{G}(\xi) = \frac{-i}{2\pi}\int_{-\infty}^{\infty} \exp\left[ik_1\xi'\right]\frac{d}{d\xi'}\log\left[\frac{H(\xi')-H(\xi)}{\xi'-\xi}\right]d\xi'. \tag{4.14}$$

For present purposes we assume that the bottom topography $h(x)$ be twice differentiable. It follows that the functions $H(\xi)$, $K(\eta)$ are also twice differentiable, and (2.9) implies

$$H'(\xi)\to \begin{matrix}\beta_L\\ \beta_R\end{matrix} \quad \text{as} \quad \xi\to\begin{matrix}-\infty\\ +\infty\end{matrix}, \tag{4.15}$$

where

$$\beta_L, \beta_R = \frac{c-\alpha_L}{c+\alpha_L}, \quad \frac{c-\alpha_R}{c+\alpha_R}. \tag{4.16}$$

There are three significant ways in which we may consider the limit $k_1\to\infty$. First, we may keep ϵ = constant, so that the amplitudes of the stream function and the pressure fluctuations (as may be seen from equations (2.1)) will be constant. The amplitudes of the fluid velocity and the fluid particle displacements, proportional to $k_1\epsilon$, will become very large in this case, as will the incident energy flux $|\overline{p\mathbf{u}}|$, being proportional to $k_1\epsilon^2$. Second, we may keep $k_1\epsilon$ constant, and third, $k_1\epsilon^2$. Each of these limits will have its own physical relevance.

In order to observe the behaviour of $\mathscr{G}(\xi)$, $\mathscr{G}'(\xi)$ as $k_1\to\infty$, it is only necessary to observe that the functions

$$\frac{d}{d\xi'}\log\left[\frac{H(\xi')-H(\xi)}{\xi'-\xi}\right], \quad \frac{d}{d\xi}\frac{d}{d\xi'}\log\left[\frac{H(\xi')-H(\xi)}{\xi'-\xi}\right]$$

are everywhere bounded and continuous functions of ξ, ξ', by virtue of the existence of the second derivative of $H(\xi)$, and that they are both

$$\sim \frac{\text{function of }\xi}{\xi'^2} \quad \text{as} \quad |\xi'|\to\infty.$$

Hence both functions are absolutely integrable and by the Riemann–Lebesgue theorem

$$\lim_{k_1\to\infty} \mathscr{G}(\xi), \mathscr{G}'(\xi) = 0. \tag{4.17}$$

It then follows directly that $F_2(\xi) = \mathscr{F}_2(\eta)$ (as the solution of (4.10)) and its derivative will similarly vanish in this limit, and its asymptotic form will in general be dependent on the nature of the bottom topography function $K(\eta)$.

The amplitudes of the stream function, velocity and energy flux associated with the back-reflected wave ψ_R are $O(\epsilon|F_2(\xi)|)$, $O(\epsilon|F_2'(\xi)|)$ and $O(\epsilon^2|F_2(\xi)F_2'(\xi)|)$ respectively, and each of these quantities will vanish as $k_1\to\infty$ in any of the three ways mentioned above. Furthermore, the ratios of their magnitudes relative to the magnitudes of the corresponding quantities in the incident wave will also vanish, in each of the three limits.†

† If there are some points on the topography where $H'(\xi)$ exists but $H''(\xi)$ does not, $\mathscr{G}(\xi)$ will still vanish in the limit $k_1\to\infty$ but $\mathscr{G}'(\xi)$ may not. The nature of the limit will depend on the local behaviour of $H(\xi)$.

Hence, in every sense, the back-reflected wave vanishes when k_1 becomes large—the incident wave feels the bottom to be locally plane, and reflects as a local plane wave. These results are therefore consistent with the WKB approximation.

The formal limit $k_1 \to 0$ is not without interest but its consideration will be discussed elsewhere because it has ramifications which are beyond the scope of the present work. One may, however, make the fairly obvious remark that if one regards the incident wave (specified by k_1, ϵ and ω) as fixed and decreases the horizontal length scale of the bottom topography (bearing in mind that its slope is limited by the 'flat bump' criterion), the back-reflected wave must vanish as the surface approaches a flat plane.

These results suggest that for a given incident wave and bottom topography satisfying the 'flat bump' and smoothness criteria specified above, the back-reflected wave is only significant if the horizontal scale of the bottom variations is $O(\lambda)$, where $\lambda = (2\pi/k_1 c)(1+c^2)^{\frac{1}{2}}$, the horizontal wavelength of the incident wave.

5. Solutions for special cases

As readers familiar with integral equations will know there is no suitable, universally applicable method available for solving equation (4.10), unless the kernel happens to have some simple form (for a discussion of the general theory see, for example, Morse & Feshbach (1953, ch. 8)). Usually one must resort to approximate methods in order to solve specific problems. In this section two examples are studied with the purpose of illustrating the features of the reflexion process and determining numerical magnitudes. Two approximate methods are used which may be applicable to a wide class of topographies.

Example 1

Remembering that the topography may be represented by an equation of the form $\xi = -K(\eta)$, we take

$$K(\eta) = \eta + \frac{2da^2}{\eta^2+a^2}, \tag{5.1}$$

and the topography represented by this equation is shown in figure 3. It may easily be seen that the condition for this bump to be a 'flat bump', i.e. $|h'(x)| < c$, is

$$|d/a| < 4/3\surd 3. \tag{5.2}$$

With $K(\eta)$ given by (5.1),

$$\frac{K(\eta')-K(\eta)}{\eta'-\eta} = 1 - \frac{2da^2(\eta'+\eta)}{(\eta'^2+a^2)(\eta^2+a^2)}, \tag{5.3}$$

and it is readily verified that the condition for this quantity to be always positive is, again, equation (5.2). Hence (4.10) becomes

$$\begin{aligned}\mathscr{F}_2(\eta) &= G(\eta) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathscr{F}_2(\eta')\frac{d}{d\eta'}\log\left[1-\frac{2da^2(\eta+\eta')}{(\eta'^2+a^2)(\eta^2+a^2)}\right]d\eta', \\ &= G(\eta) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mathscr{F}_2(\eta')}{d\eta'}\log\left[1-\frac{2da^2(\eta+\eta')}{(\eta'^2+a^2)(\eta^2+a^2)}\right]d\eta'.\end{aligned} \tag{5.4}$$

Expanding the logarithm and integrating term by term yields

$$\mathscr{F}_2(\eta) = G(\eta) + \frac{2a^2d(c_{10}+c_{11}\eta)}{\eta^2+a^2} + \frac{2a^4d^2(c_{20}+c_{21}\eta+c_{22}\eta^2)}{(\eta^2+a^2)^2} + \dots + \sum_{m=0}^{n} \frac{c_{nm}\eta^m(2a^2d)^n}{(\eta^2+a^2)^n} + \dots, \tag{5.5}$$

where the c_{nm}'s are constants, the first five being given by

$$c_{10} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mathscr{F}_2(\eta')}{d\eta'}\frac{\eta' d\eta'}{(\eta'^2+a^2)}, \quad c_{11} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mathscr{F}_2(\eta')}{d\eta'}\frac{d\eta'}{\eta'^2+a^2},$$

$$c_{20} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mathscr{F}_2(\eta')}{d\eta'}\frac{\eta'^2 d\eta'}{(\eta'^2+a^2)^2}, \quad c_{21} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mathscr{F}_2(\eta')}{d\eta'}\frac{2\eta' d\eta'}{(\eta'^2+a^2)^2},$$

$$c_{22} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mathscr{F}_2(\eta')}{d\eta'}\frac{d\eta'}{(\eta'^2+a^2)^2}. \tag{5.6}$$

The series (5.5) will converge provided the inequality (5.2) is satisfied.

FIGURE 3. Form of the bottom topography for example 1. (a) $d > 0$, (b) $d < 0$.

The next step is to evaluate the function $G(\eta)$. From equation (4.9) and the Plemelj formulae of complex analysis, $G(\eta)$ may be expressed as

$$G(\eta) = -\frac{1}{2}\left[\frac{1}{\pi i}\mathrm{P}\int_{-\infty}^{\infty} \frac{\exp[-ik_1K(\eta')]}{\eta'-\eta}d\eta' + \exp[-ik_1K(\eta')]\right], \tag{5.7}$$

$$= -\lim_{z_0\to\eta+} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\exp[-ik_1K(\eta')]}{\eta'-z_0}d\eta', \tag{5.8}$$

where $z_0 \to \eta+$ implies that $z_0 \to \eta$ but always with a positive imaginary part.

$$G(\eta) = -\lim_{z_0\to\eta+} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\exp[-ik_1\eta']}{\eta'-z_0}\exp[-ik_1 2a^2d/(\eta'^2+a^2)]d\eta'. \tag{5.9}$$

The integrand of the complex η' plane is single-valued and exponentially small in the lower half-plane, but is complicated by the essential singularity at $\eta' = -ia$ (see figure 4). However, we may expand the exponential term in its power series, obtaining

$$\exp[-ik_1 2a^2d/(\eta'^2+a^2)] = 1 - \frac{ik_1 2a^2 d}{\eta'^2+a^2} - \frac{(2k_1d)^2a^4}{2(\eta'^2+a^2)^2} + \dots, \tag{5.10}$$

and integrate term by term. Closing the contour in the lower half-plane shows that the first term is zero, so that

$$G(\eta) = \lim_{z_0\to\eta+} \frac{k_1da^2}{\pi}\int_{-\infty}^{\infty} \frac{\exp[-ik_1\eta']d\eta'}{(\eta'^2+a^2)(\eta'-z_0)} - \lim_{z_0\to\eta+} \frac{i(k_1d)^2a^4}{\pi}\int_{-\infty}^{\infty} \frac{\exp[-ik_1\eta']d\eta'}{(\eta'^2+a^2)(\eta'-z_0)} + \dots. \tag{5.11}$$

This series will converge for all values of $k_1 d$ by virtue of the factorial sum in the denominator, so that any desired accuracy may be obtained by taking a sufficient number of terms. If $k_1 d > 1$ several terms must be taken, and this would be a laborious process, so we will assume $k_1 d < 1$ and be satisfied with the first one or two terms. Evaluating the residues at $\eta = -ia$ and then taking the simple limit $z_0 \to \eta +$ yields

$$G(\eta) = -k_1 d \exp[-k_1 a] \frac{a(\eta - ia)}{\eta^2 + a^2}\left[1 - ik_1 d(1 + k_1 a) + \frac{k_1 da(\eta - ia)}{\eta^2 + a^2} + O(k_1 d)^2\right], \tag{5.12}$$

neglecting terms of order $(k_1 d)^3$. However, these neglected terms show a similar shape and form to those retained, and therefore their inclusion, even if they were important, should not alter the character of $G(\eta)$ very much.

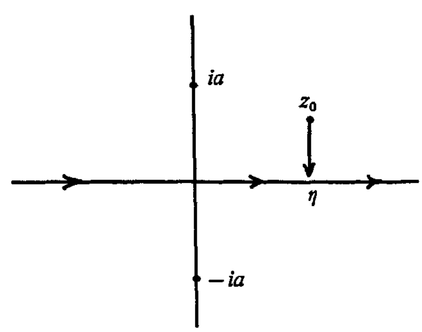


FIGURE 4. The complex η' plane for $G(\eta)$

From (5.5), (5.6) and (5.12) we may now obtain an approximate solution for $\mathscr{F}_2(\eta)$. The higher-order terms of the series (5.5) are essentially of order $(d/a)^n$. One may terminate the series to the accuracy required, and then substitute for $\mathscr{F}_2(\eta)$ in the expressions (5.6) for the appropriate constants c_{nm}, obtaining a set of linear equations which may be solved for these coefficients. For the purpose of illustrating the character of the solution we truncate the series for $\mathscr{F}_2(\eta)$ at the second term and write

$$\mathscr{F}_2(\eta) = G(\eta) + \frac{2a^2 d(c_{10} + c_{11}\eta)}{\eta^2 + a^2}, \tag{5.13}$$

where $G(\eta)$ is given by (5.12). Equation (5.6) then yields

$$\left.\begin{aligned} c_{10} &= -\frac{k_1 d \exp[-k_1 a]}{8a(1 + (d/4a)^2)}[1 + (id/4a)](1 - ik_1 d(\tfrac{3}{2} + k_1 a)), \\ c_{11} &= \frac{ik_1 d \exp[-k_1 a]}{8a^2(1 + (d/4a)^2)}[1 - (id/4a)](1 - ik_1 d(\tfrac{3}{2} + k_1 a)), \end{aligned}\right\} \tag{5.14}$$

and from (5.13) we have

$$\mathscr{F}_2(\eta) = \frac{c_1((\eta/a) - i)}{(\eta/a)^2 + 1} + \frac{c_2((\eta/a) + i)}{(\eta/a)^2 + 1} + \frac{c_3((\eta/a) - i)^2}{((\eta/a)^2 + 1)^2} + O((d/a)^3, (k_1 d)^3), \tag{5.15}$$

where

$$\left.\begin{aligned} c_1 &= -k_1 d \exp[-k_1 a]\left[1 - ik_1 d(1+k_1 a) - (d/a)^2 \frac{(1 - ik_1 d(\frac{3}{2} + k_1 a))}{16(1+(d/4a)^2)}\right], \\ c_2 &= \frac{ik_1 d \exp[-k_1 a](d/a)}{4(1+(d/4a)^2)}[1 - ik_1 d(\tfrac{3}{2} + k_1 a)], \quad c_3 = -(k_1 d)^2 \exp[-k_1 a]. \end{aligned}\right\} \tag{5.16}$$

This expression will be a good approximation to $\mathscr{F}_2(\eta)$ provided $k_1 d \ll 1$, $d/a \ll 1$, and indeed a simpler approximation for this case would be

$$F_2(\xi) = \mathscr{F}_2(\eta) \sim -k_1 d \exp[-k_1 a] \frac{(\eta/a) - i}{(\eta/a)^2 + 1}. \tag{5.17}$$

We may notice several things about this solution. First, the back-reflected wave $F_2(\xi)$ and the correction $\mathscr{F}_2(\eta)$ to $\exp[-ik_1 K(\eta)]$ for the onward transmitted wave $F_1(\eta)$ have magnitudes which resemble the shape of the bump (although the wave is somewhat broader than the bump, behaving like $\eta/(\eta^2 + a^2)$ for η large rather than $1/(\eta^2 + a^2)$). Second, the leading term in the expression for $F_2(\xi)\exp[-i\omega t]$ (equation (5.17)) has phase propagation in the direction of ξ decreasing, as might be expected, and has the form of a single (i.e. 'one-wave-length') wave, characteristic of the bump or bottom topography, with a 'wave-length' of order a. Also, the leading term for $\mathscr{F}_2(\eta)\exp[-i\omega t]$ (again equation (5.17)) has phase propagation in the direction appropriate for incoming plane waves,† and this, one may expect, is present in order to partly cancel out the incoming part of $\exp[-ik_1 K(\eta)]\exp[-i\omega t]$. Third, the above equations indicate that the back-reflected wave energy will be a maximum when $k_1 d$, $k_1 a = O(1)$, and that it may be a significant proportion of the energy incident on the bump (e.g. 10–20 %).

Example 2

From the classical Neumann theory of integral equations, the general solution of (4.10) has the form

$$\mathscr{F}_2(\eta) = G(\eta) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(\eta')\mathscr{K}(\eta,\eta')\,d\eta', \tag{5.18}$$

where $\mathscr{K}(\eta,\eta') = K_1(\eta,\eta') + K_2(\eta,\eta') + K_3(\eta,\eta') + \ldots,$

$$\left.\begin{aligned} K_1(\eta,\eta') &= \frac{d}{d\eta'}\log\left[\frac{K(\eta') - K(\eta)}{\eta' - \eta}\right], \\ K_n(\eta,\eta') &= \frac{1}{(2\pi i)^{n-1}}\int_{-\infty}^{\infty} K_1(\eta,\eta'')K_{n-1}(\eta'',\eta')\,d\eta'' \quad (n > 1). \end{aligned}\right\} \tag{5.19}$$

The convergence of this series suggests that a good approximate solution is

$$\mathscr{F}_2(\eta)_B = G(\eta) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(\eta')\frac{d}{d\eta'}\log\left[\frac{K(\eta') - K(\eta)}{\eta' - \eta}\right]d\eta'. \tag{5.20}$$

This is equivalent to the Born approximation of classical scattering theory (or perhaps, more accurately, the Kirchoff approximation since this is a surface

† This may be seen by expressing $\mathscr{F}_2(\eta)\exp[-i\omega t]$ in the form $r\exp[i(\phi - \omega t)]$, using (5.17).

scatterer (see Morse & Feshbach 1953, vol. II, §9.2); however we will refer to it as the Born approximation here). The accuracy of this procedure will depend in general on the topography involved, and the extra accuracy gained by including the next term in the series may be gauged by estimating its magnitude directly. Evaluating these integrals will not, however, be a simple process for most topographies.

As an illustration of this procedure we consider the bottom topography represented by

$$K(\eta) = \eta - (2d/\pi)\arctan \eta/a. \tag{5.21}$$

This represents a smooth change in depth from one horizontal plane to another, with a difference d in depth. The necessary condition for the topography to satisfy the flat bump criterion is

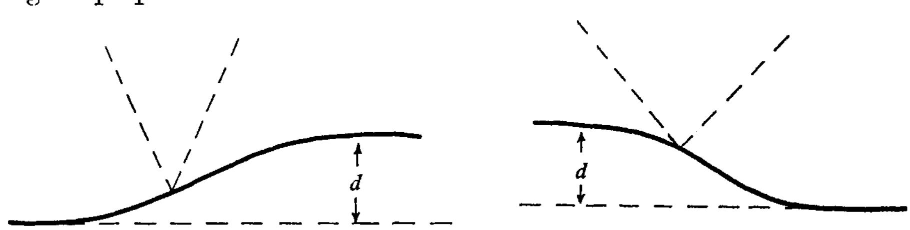
$$|d/a| < \tfrac{1}{2}\pi, \tag{5.22}$$

and the topography is illustrated in figure 5. The function $G(\eta)$ will then be given by

$$G(\eta) = -\lim_{z_0\to\eta+} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\exp[-ik_1\eta']\exp[(i2dk_1/\pi)\arctan(\eta'/a)]}{\eta'-z_0}\,d\eta', \tag{5.23}$$

$$= -\lim_{z_0\to\eta+} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\exp[-ik_1\eta']}{\eta'-z_0}\left(\frac{ia-\eta'}{ia+\eta'}\right)^{k_1 d/\pi} d\eta', \tag{5.24}$$

using the properties of the arctan function.



(*a*) (*b*)

FIGURE 5. Form of the bottom topography for example 2. (*a*) $d > 0$, (*b*) $d < 0$.

If $d < 0$ (down-sloping topography), after some manipulation this may be written as

$$G(\eta) = -\sin\gamma\pi\exp[-k_1 a]\int_0^{\infty} \frac{\exp[-k_1 y]}{a-i\eta+y}\left(\frac{y}{2a+y}\right)^{\gamma} dy, \tag{5.25}$$

where $\gamma = k_1|d|/\pi$. It is interesting to note that $G(\eta)$ vanishes when γ is an integer, implying that $F_2(\xi)$, $\mathscr{F}_2(\eta)$ vanish for these cases and

$$F_1(\eta) = \exp[-ik_1 K(\eta)]$$

is the appropriate onward-transmitted wave. Also, $G(\eta)$ is an integral over y of terms of the form

$$\frac{\eta - i(a+y)}{\eta^2 + (a+y)^2}$$

(cf. (5.17)), which have their phase propagation in the opposite direction to that of outward-going plane waves.

When $d > 0$ (upward-sloping topography) and γ is an integer, $G(\eta)$ does not vanish but may be evaluated directly from equation (5.24) by closing the contour in the lower half-plane. For $k_1 d = \pi$ we obtain

$$G(\eta) = -2ia \exp[-k_1 a] \frac{\eta - ia}{\eta^2 + a^2}, \tag{5.26}$$

for all values of a. This term again represents the same phase-propagation as (5.17).

The Born approximation for this case is given by

$$\mathscr{F}_2(\eta) = G(\eta) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(\eta') \frac{d}{d\eta'} \log\left[1 - \frac{2d \arctan[a(\eta'-\eta)/(\eta\eta'+a^2)]}{\pi(\eta'-\eta)}\right] d\eta'. \tag{5.27}$$

The analytic evaluation of this second integral, even in this simple case, appears to be very difficult. However, it seems fairly clear from the nature of the first term, $G(\eta)$, that the broad features of the solution will be much the same as those of example 1, the only major difference being the vanishing (and corresponding oscillation in intensity) of the back-reflected wave for some values of $k_1 d$ for the down-sloping case.

6. The linearized boundary condition

There is one class of topographies where good approximations in general may be obtained for the reflected waves with considerably less effort than is required above, and this is the case where the bottom topography variations may be regarded as small perturbations of a plane surface. Using the condition that the component of fluid velocity normal to the actual surface must vanish at the boundary, one may regard the interaction between the flow which would be present without the boundary perturbations with the perturbations themselves as a new source of fluid motion, situated on the plane surface. This will generate internal waves which must satisfy the radiation condition. This has been done by Cox & Sandstrom (1962) and Hendershott (1966), who considered long surface waves interacting with bottom topography to generate internal waves. For the problem under consideration here, namely the interaction of internal waves with bottom topography to generate more internal waves, we shall start with the integral equation of §4, which is a more systematic procedure than the direct approach described above, although they both give the same results.

Let the bottom topography have the equation

$$z = \alpha x + f(x), \quad (|\alpha| < c), \tag{6.1}$$

where

$$\left.\begin{aligned} |f(x)| \leqslant d, \quad |f'(x)| = O(d/a) = O(\delta), \\ \delta = d/a \ll 1. \end{aligned}\right\} \tag{6.2}$$

a is the shortest horizontal length scale of the variation of $f(x)$. In the characteristic variables (6.1) has the form

$$\xi = -[(c+\alpha)/(c-\alpha)]\eta - L(\eta), \tag{6.3}$$

where $L(\eta)$ may be expressed

$$L(\eta) = -\frac{2c}{c-\alpha} f\left(\frac{-\eta}{c-\alpha}\right)\left[1 + \frac{1}{c-\alpha} f'\left(\frac{-\eta}{c-\alpha}\right) + O(\delta^2)\right], \tag{6.4}$$

so that $L(\eta) = O(d)$ $(c/(c-\alpha)$ is assumed not to be large). From (4.10) the function $\mathscr{F}_2(\eta)$ is given by

$$\mathscr{F}_2(\eta) = G(\eta) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathscr{F}_2(\eta')\frac{d}{d\eta'}\log\left[\frac{c+\alpha}{c-\alpha} + \frac{L(\eta')-L(\eta)}{\eta'-\eta}\right]d\eta', \qquad (6.5)$$

where (equation (5.8))

$$G(\eta) = -\lim_{z_0\to\eta+}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\exp\left[-ik_1((c+\alpha)/(c-\alpha))\eta'\right]\exp\left[-ik_1L(\eta')\right]}{\eta'-z_0}d\eta'. \qquad (6.6)$$

We now make a second assumption, namely that

$$\Delta = k_1 d \ll 1, \qquad (6.7)$$

which together with $\delta = d/a \ll 1$ implies that the normal scale (i.e. normal to the plane) of the bottom topography variations is much less than the incident wavelength and the tangential length scale. We also assume that $\Delta \sim \delta$. Since $k_1L(\eta) = O(\Delta)$ we have

$$\exp\left[-ik_1L(\eta')\right] = 1 - ik_1L(\eta') - \tfrac{1}{2}(k_1L(\eta'))^2 + O(\Delta^3), \qquad (6.8)$$

so that

$$G(\eta) = \underbrace{\lim_{z_0\to\eta+}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp\left[-ik_1((c+\alpha)/(c-\alpha))\eta'\right]k_1L(\eta')\,d\eta'}{\eta'-z_0}}_{O(\Delta)}$$
$$-\underbrace{\lim_{z_0\to\eta+}\frac{i}{4\pi}\int_{-\infty}^{\infty}\frac{\exp\left[-ik_1((c+\alpha)/(c-\alpha))\eta'\right]k_1^2L^2(\eta')\,d\eta'}{\eta'-z_0}}_{O(\Delta^2)} + O(\Delta^3). \qquad (6.9)$$

It is readily shown that the maximum value of $[L(\eta')-L(\eta)]/(\eta'-\eta)$ anywhere is $|dL/d\eta|$, so that

$$\frac{L(\eta')-L(\eta)}{\eta'-\eta} = O(\delta), \qquad (6.10)$$

for all η', η. Expanding the logarithm in (6.5) and integrating by parts then yields

$$\mathscr{F}_2(\eta) = G(\eta) - \frac{(c-\alpha)/(c+\alpha)}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mathscr{F}_2(\eta')}{d\eta'}\left(\frac{L(\eta')-L(\eta)}{\eta'-\eta}\right)d\eta' + O(\Delta\delta^2), \qquad (6.11)$$

and to second order in Δ, δ we have

$$\mathscr{F}_2(\eta) = G(\eta) - \frac{(c-\alpha)/(c+\alpha)}{2\pi i}\int_{-\infty}^{\infty}\frac{dG(\eta')}{d\eta'}\left(\frac{L(\eta')-L(\eta)}{\eta'-\eta}\right)d\eta' + O(\Delta^3, \Delta\delta^2). \qquad (6.12)$$

In fact, to calculate $\mathscr{F}_2(\eta)$ to first order, only the first term in (6.9) is required, and successive higher-order approximations may be readily obtained if desired.

We now investigate the reflexion from a sinusoidal bump of the form

$$f(x) = d\cos lx, \quad ld \ll 1. \qquad (6.13)$$

(For this case the solution for any acceptable function $f(x)$ may be obtained by Fourier superposition of the results for its Fourier components.) From (6.4) we obtain

$$L(\eta) = -\frac{2cd}{c-\alpha}\cos\frac{l\eta}{c-\alpha} + \frac{2c^2d^2l}{(c-\alpha)^3}\sin\frac{2l\eta}{c-\alpha} + O(d\delta^2), \qquad (6.14)$$

and thence from (6.9)

$$\begin{aligned} G(\eta) &= 0+O(\Delta^2) \quad (l < k_1(c+\alpha)), \\ &= -\frac{ick_1d}{c-\alpha}\exp\left[\frac{i\eta}{c-\alpha}(l-k_1(c+\alpha))\right]+O(\Delta^2) \quad (l > k_1(c+\alpha)), \end{aligned} \tag{6.15}$$

which is also the form of $\mathscr{F}_2(\eta)$ to first order in Δ, δ. From equations (4.11), (6.12) we finally have

$$\left.\begin{aligned} &\left.\begin{aligned} F_2(\xi) &= 0+O(\Delta^2,\Delta\delta), \\ F_1(\eta) &= \left[1-\left(\frac{ck_1d}{c-\alpha}\right)^2\right]\exp\left[-i\left(\frac{c+\alpha}{c-\alpha}\right)k_1\eta\right] \\ &\quad +\frac{ick_1d}{c-\alpha}\left(\exp\left[-\frac{i\eta}{c-\alpha}(k_1(c+\alpha)+l)\right]\right. \\ &\quad \left.+\exp\left[-\frac{i\eta}{c-\alpha}(k_1(c+\alpha)-l)\right]\right)+O(\Delta^2,\delta^2), \end{aligned}\right\} \quad l < k_1(c+\alpha), \\ &\left.\begin{aligned} F_2(\xi) &= -\frac{ick_1d}{c-\alpha}\exp\left[-\frac{i\xi}{c+\alpha}(l-k_1(c+\alpha))\right]+O(\Delta^2,\delta^2), \\ F_1(\eta) &= \left[1-\frac{c^2k_1d.ld}{(c-\alpha)^2(c+\alpha)}\right]\exp\left[-i\left(\frac{c+\alpha}{c-\alpha}\right)k_1\eta\right] \\ &\quad +\frac{ick_1d}{c-\alpha}\exp\left[-\frac{i\eta}{c-\alpha}(k_1(c+\alpha)+l)\right]+O(\Delta^2,\delta^2), \end{aligned}\right\} \quad l > k_1(c+\alpha). \end{aligned}\right\} \tag{6.16}$$

The second-order part of the $\exp[-i((c+\alpha)/(c-\alpha))k_1\eta]$ term has been included because it is needed in order to calculate the energy flux to the lowest order. The instantaneous energy flux across any surface S is given by

$$\int p\mathbf{u}.d\mathbf{S},$$

and for a plane wave given by

$$\psi = \epsilon\exp[i(k_1\zeta-\omega t)], \quad \zeta = \pm\xi \quad \text{or} \quad \eta,$$

the time-averaged energy flux (per unit area) is given by

$$E_{fl} = \epsilon^2k_1\rho_0[(1/\omega)(\omega^2-f^2)(N^2-\omega^2)]^{\frac{1}{2}}, \tag{6.17}$$

in the appropriate direction. The back-reflected energy flux is then

$$\left.\begin{aligned} E_{flR} &= \epsilon^2\rho_0\left(\frac{ck_1d}{c-\alpha}\right)^2\left[\frac{l}{c+\alpha}-k_1\right]\left[\frac{1}{\omega}(\omega^2-f^2)(N^2-\omega^2)\right]^{\frac{1}{2}} \quad (l > k_1(c+\alpha)), \\ &= 0 \qquad\qquad (l < k_1(c+\alpha)), \end{aligned}\right\} \tag{6.18}$$

to second order in Δ, δ.

A single plane wave incident on a small amplitude sinusoidal bottom therefore produces, in addition to the basic wave reflected from the plane surface, two new waves whose wave-numbers are the sum and difference of those of the reflected (or incident, as appropriate) wave and the bottom topography, projected in the appropriate characteristic direction. The 'sum' wave is always onward-transmitted, whereas the 'difference' wave is back-reflected when the incident wavelength is longer than that of the (projected) bottom topography. At the changeover point, $l = k_1(c+\alpha)$, this 'difference' wave has infinite wavelength and the energy flux associated with it vanishes. The back-reflected wavelength tends to be long, and is always longer than the projected wavelength of the sinusoidal bottom. The back-reflected wave is not wholly absent when $l < k_1(c+\alpha)$, as it appears for smaller l at higher orders. For example, at second order in Δ for ψ the change-over occurs at

$$l = \tfrac{1}{2}k_1(c+\alpha). \tag{6.19}$$

It is interesting also to note that the back-reflected energy flux is proportional to $k_1 d . ld = \Delta\delta$, which increases with d as d^2, indicating that this linearized theory, although instructive and quite possibly very useful, is really on the fringes of the phenomenon.

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REFERENCES

Baines, P. G. 1969 Ph.D. Thesis, University of Cambridge.

Barcilon, V. & Bleistein, N. 1969*a* Scattering of inertial waves in a rotating fluid. *Studies in Appl. Math.* **48**, 91.

Barcilon, V. & Bleistein, N. 1969*b* Scattering of inertial waves by smooth convex cylinders. *Studies in Appl. Math.* **48**, 351.

Carrier, G. F., Krook, M. & Pearson, C. E. 1966 *Functions of a Complex Variable*. McGraw-Hill.

Cox, C. S. & Sandstrom, H. 1962 Coupling of surface and internal waves in water of variable depth. *Oceanog. Soc. Japan, 20th Anniv. Vol.*

Eckart, C. 1960 *Hydrodynamics of Oceans and Atmospheres*. Pergamon.

Greenspan, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.

Hendershott, M. 1968 Inertial oscillations of tidal period. *Scripps Institution Oceanog. Report* Ref. 68–12.

Hurley, D. G. 1970 *J. Fluid Mech.* **43**, 97.

Keller, J. B. & Mow, Van C. 1969 Internal wave propagation in an inhomogeneous fluid of non-uniform depth. *J. Fluid Mech.* **38**, 365.

Lamb, H. 1945 *Hydrodynamics*. Dover.

Larsen, L. H. 1969 Internal waves incident on a knife edge barrier. *Deep Sea Res.* **16**, 411.

Lighthill, M. J. 1958 *Fourier Analysis and Generalized Functions*. Cambridge University Press.

Lighthill, M. J. 1965 Group velocity. *J. Inst. Math. Applics.* **1**, 1.

Lighthill, M. J. 1967 On waves generated in dispersive systems by travelling forcing effects, with applications to the dynamics of rotating fluids. *J. Fluid Mech.* **27**, 725.

Longuet-Higgins, M. S. 1969 On the reflexion of wave characteristics from rough surfaces. *J. Fluid Mech.* **37**, 231.

Morse, P. M. & Feshbach, H. 1953 *Methods of Theoretical Physics*, Parts I and II. McGraw-Hill.

Phillips, O. M. 1963 Energy transfer in rotating fluids by reflection of inertial waves. *Phys. Fluids*, **6**, 513.

Phillips, O. M. 1966 *Dynamics of the Upper Ocean*. Cambridge University Press.

Rattray, M., Dworski, J. G. & Kovala, P. E. 1969 Generation of long internal waves at the continental slope. *Deep Sea Res.* **16** (Supplement), 179.

Robinson, R. M. 1969 The effects of a vertical barrier on internal waves. *Deep Sea Res.* **16**, 421.

Sandstrom, H. 1966 Ph.D. Thesis, University of California at San Diego.